

Derivation of Non-isotropic Phase Equations from a General Reaction-Diffusion Equation

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Abstract

A non-isotropic version of phase equations such as the Burgers equation, the K-dV-Burgers equation, the Kuramoto-Sivashinsky equation and the Benney equation in the three-dimensional space is systematically derived from a general reaction-diffusion system by means of the renormalization group method.

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1 Introduction

It was a long time ago that the Kuramoto-Sivashinsky (K-S) equation was proposed as a higher-order phase equation describing an unstable phase state [1]. Although there have been many works based on the K-S equation since then, there are only few derivations of the K-S equation based on the singular perturbation methods. Recently, the isotropic K-S equation has been derived as a phase equation of a periodically oscillating solution of the complex Ginzburg-Landau equation by means of the renormalization group method [2]. In a reaction-diffusion system, spatial symmetry often breaks so that both spatially and temporally oscillating solutions emerge. In such a case, it is anticipated that a slowly varying phase of the symmetry-breaking oscillating state is asymptotically governed by a non-isotropic phase equation. The Gross-Newell's phase equation was derived by the RG method [3]. However, there are no explicit derivations of such non-isotropic phase equations as the non-isotropic K-S equation, the K-dV-Burgers equation and the Benney equation [4] from a general reaction-diffusion equation. In this paper, we derive such non-isotropic phase equations to a symmetry-breaking state of a general reaction-diffusion equation by means of the renormalization group (RG) method.

2 Renormalization Group Method

The perturbative RG method introduced in [5] is shown to be interpreted as the procedure to obtain an asymptotic expression of a generator of a renormalization transformation based on the Lie group [6]. This Lie group approach provides the following simple recipe for obtaining an asymptotic form of a RG equation from ordinary differential equations (ODE).

- (1) Get a secular series solution of a perturbed equation by means of naive perturbation calculations.
- (2) Find integral constants, which are renormalized to eliminate all the secular terms in the perturbed solution and give a renormalization transformation for the integral constants.
- (3) Rewrite the renormalization transformation by excuting an arbitrary shift operation on the independent variable: $t \rightarrow t + \tau$ and derive a representation of a Lie group underlying the renormalization transformation.
- (4) By differentiating the representation of the Lie group with respect to arbitrary τ , we obtain an asymptotic expression of the generator, which yields an

asymptotic RG equation.

This procedure is valid for general ODE regardless of translational symmetry. The above recipe for ODE is also applicable to autonomous partial differential equations (PDE) by choosing suitable polynomial kernels of the linearized operator appearing in perturbed equations. First, we should take the lowest-order polynomial, of which degree is one, as the leading order secular term. As perturbation calculations proceed to the higher order, polynomial kernels of higher degrees are included in the higher-order secular terms order by order. When we can continue this process consistently, we say that secular terms of polynomials are renormalizable or, simply, the consistent renormalization condition is satisfied in the sense of the Lie approach. If this is the case, we can determine suitable polynomial kernels among infinite number of kernels of the linearized operator and the step (1) in the recipe is completed. There are no problems in the other steps. Thus, the Lie-group approach is consistently applicable to PDE and derivation of some soliton equations and simple phase equations (e.g. the isotropic Burgers equation) was presented in [6]. In the following sections, using this Lie-group approach of the RG method, we derive various non-isotropic phase equations from a general reaction diffusion system.

3 Non-isotropic Burgers Equation

Let us consider a general reaction-diffusion system of equations:

$$\partial_t U = F(U) + D\nabla^2 U, \quad (3.1)$$

where U is an n -dimensional vector and D is an $n \times n$ constant matrix. Suppose (3.1) has a spatially and temporally oscillating solution $U_0 = U_0(k, kx - \omega(k)t + \phi)$ satisfying

$$-(\omega U_{0,\theta} + F(U_0) + k^2 D U_{0,\theta\theta}) = 0, \quad (3.2)$$

where k and ϕ are arbitrary constants; $\omega(k)$ is a definite function of k ; $\theta = kx - \omega t + \phi$; the suffix θ denotes the derivative with respect to θ and U_0 is a 2π periodic function of θ . For later convenience, we list some useful identities. Differentiating (3.2) with respect to θ and k , we have

$$L(\theta)U_{0,\theta} \equiv -(\omega\partial_\theta + F'(U_0) \cdot + k^2 D \partial_\theta^2)U_{0,\theta} = 0, \quad (3.3)$$

$$L U_{0,\theta\theta} = F'' : U_{0,\theta}^2, \quad (3.4)$$

$$L U_{0,k} = (\dot{\omega} + 2k D \partial_\theta)U_{0,\theta} \equiv M U_{0,\theta}, \quad (3.5)$$

$$L U_{0,kk} = F'' : (U_{0,k})^2 + 2M U_{0,\theta k} + 2D U_{0,\theta\theta} + \ddot{\omega} U_{0,\theta}, \quad (3.6)$$

$$L U_{0,\theta k} = M U_{0,\theta\theta} + F'' : U_{0,\theta} U_{0,k}, \quad (3.7)$$

where $F' \cdot V = (V \cdot \nabla_U)F(U)|_{U=U_0}$, $F'' : VW = [(W \cdot \nabla_U)(V \cdot \nabla_U)F(U)]|_{U=U_0}$, $\dot{\omega} = \partial_k \omega$ and $\ddot{\omega} = \partial_k^2 \omega$.

Let us seek a secular solution close to $U_0(k, \theta)$:

$$U = U_0(k + \kappa(x, \mathbf{r}_\perp, t), \theta + \delta(x, \mathbf{r}_\perp, t)) + \tilde{U}(\theta, x, \mathbf{r}_\perp, t), \quad (3.8)$$

where $\mathbf{r}_\perp = (0, y, z)$; $\delta(x, \mathbf{r}_\perp, t)$ and $\kappa(x, \mathbf{r}_\perp, t)$ are small secular deviations from the constant phase ϕ and the wavenumber k respectively and so

$$\partial_x \delta \equiv \delta_x = \kappa. \quad (3.9)$$

$\tilde{U}(\theta, x, \mathbf{r}_\perp, t)$ represents small perturbed fields which modifies the 0-th order field pattern U_0 and is not expressed by differentials of $U_0(k, \theta)$. The arguments (x, \mathbf{r}_\perp, t) designate “secular variables”, that is, all secular perturbed fields are polynomials with respect to (x, \mathbf{r}_\perp, t) and periodic with θ . An expansion of U_0 in terms of small deviations δ and κ yields

$$\begin{aligned} U_0(k + \kappa, \theta + \delta) &= U_0(k, \theta) + \delta U_{0,\theta} + \kappa U_{0,k} \\ &+ (\delta^2/2)U_{0,\theta\theta} + (\kappa^2/2)U_{0,kk} + \delta\kappa U_{0,\theta k} + \dots \end{aligned}$$

Note that $\delta(x, \mathbf{r}_\perp, t)$ (and κ) is a secular or polynomial function of (x, \mathbf{r}_\perp, t) , which should be eliminated by renormalizing the phase ϕ later.

In this section we suppose that δ and \tilde{U} are expanded in terms of a small perturbation parameter ϵ as

$$\delta = \epsilon(P_1 + \epsilon P_2 + \dots), \quad \kappa = \epsilon(P_{1,x} + \epsilon P_{2,x} + \dots) \quad (3.10)$$

$$\tilde{U} = \epsilon^2(\tilde{U}_2(\theta) + \epsilon \tilde{U}_3 + \dots), \quad (3.11)$$

where the suffix x denotes the derivative with respect to x and P_j ($j = 1, 2, \dots$) are polynomials of (x, \mathbf{r}_\perp, t) , which have increasing degrees with j so that polynomial secular terms are renormalizable in the sense of the Lie approach of the RG method [6]. \tilde{U}_2 depends only on θ since the leading order term of \tilde{U} is not contain secular terms of (x, \mathbf{r}_\perp, t) ; otherwise they would not be eliminated by the RG procedure (see (4.58) and (4.59) or (5.90) and (5.91)). Then (3.8) reads

$$U = U_0(k, \theta) + \epsilon U_1 + \epsilon^2 U_2 + \dots, \quad (3.12)$$

$$U_1 = P_1 U_{0,\theta}(k, \theta) + P_{1,x} U_{0,k}(k, \theta), \quad (3.13)$$

$$\begin{aligned} U_2 &= P_2 U_{0,\theta} + P_{2,x} U_{0,k} + (1/2)P_1^2 U_{0,\theta\theta} \\ &+ P_1 P_{1,x} U_{0,\theta k} + (1/2)P_{1,x}^2 U_{0,kk} + \tilde{U}_2. \end{aligned} \quad (3.14)$$

Introducing the Galilean transformation

$$x' = x - \dot{\omega}t, \quad t' = t, \quad (3.15)$$

and substituting (3.12) into (3.1), we have to the first order perturbed field U_1

$$(\partial_{t'} + L(\theta) - \partial_{x'} M - \nabla^2 D)U_1 \equiv \tilde{L}U_1 = 0, \quad (3.16)$$

where $(\theta, x', t', \mathbf{r}_\perp)$ is considered as a set of independent variables. Hereafter, the prime attached to (x', t') is omitted for simplicity. Substituting (3.13) into (3.16) and using (3.3) and (3.5), we have

$$\begin{aligned} & -P_{1,xx}MU_{0,k} - \nabla^2 P_1 DU_{0,\theta} - \nabla^2 P_{1,x} DU_{0,k} \\ & + P_{1,t}U_{0,\theta} + P_{1,xt}U_{0,k} = 0. \end{aligned} \quad (3.17)$$

Since P_1 is a polynomial of (x, \mathbf{r}_\perp, t) and U_0 is a periodic function of θ , (3.17) reads

$$P_{1,xx} = 0, \quad \nabla^2 P_1 = 0, \quad \nabla^2 P_{1,x} = 0, \quad P_{1,t} = 0, \quad P_{1,xt} = 0, \quad (3.18)$$

where the suffix t denotes the derivative with respect to t . Noting that the leading order secular term consists of a polynomial of degree one, (3.18) yields

$$P_1 = P_{1,x}x + \nabla_\perp P_1 \cdot \mathbf{r}_\perp. \quad (3.19)$$

where $\nabla_\perp \equiv (0, \partial_y, \partial_z)$; $P_{1,x}$ and $\nabla_\perp P_1$ are arbitrary constants. The second order equation obeys

$$\tilde{L}U_2 = (1/2)F'' : U_1^2(\theta), \quad (3.20)$$

Substituting (3.14) into (3.20), we obtain with the aid of (3.3)–(3.7)

$$\begin{aligned} L\tilde{U}_2 &= |\nabla_\perp P_1|^2 DU_{0,\theta\theta}(\theta) - (1/2)P_{1,x}^2 \ddot{U}_{0,\theta}(\theta) + P_{2,xx}MU_{0,k}(\theta) \\ &+ \nabla^2 P_2 DU_{0,\theta}(\theta) - P_{2,t}U_{0,\theta}(\theta) - P_{2,x,t}U_{0,k}(\theta), \end{aligned} \quad (3.21)$$

which is an equation for periodic \tilde{U}_2 . Since \tilde{U}_2 is a function of θ only, all the coefficients of functions of θ in the right hand side (RHS) of (3.21) do not depend on (x, \mathbf{r}_\perp, t) , that is,

$$P_{2,xx} = c_1, \quad \nabla^2 P_2 = c_2, \quad P_{2,t} = c_3, \quad P_{2,xt} = c_4, \quad (3.22)$$

where c_1, c_2 and c_3 are non-zero constants while $c_4 = 0$ due to the consistent renormalization condition, i.e. both t and xt do not enter in P_2 as secular terms to be removed consistently by renormalization. This requirement holds through this paper. An explicit form of P_2 is given as a polynomial of degree two with respect to (x, \mathbf{r}_\perp) :

$$P_2 = P_{2,xx}x^2/2 + (\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp)P_2/2 + x(\mathbf{r}_\perp : \nabla_\perp)P_{2,x} + P_{2,t}t, \quad (3.23)$$

where all the coefficients of monomials, i.e., $P_{2,xx}$, $\nabla_\perp \nabla_\perp P_2$, $\nabla_\perp P_{2,x}$ and $P_{2,t}$ are arbitrary constants and

$$(\overbrace{\mathbf{r}_\perp \cdots \mathbf{r}_\perp}^n : \overbrace{\nabla_\perp \cdots \nabla_\perp}^n) \equiv \sum_{k=0}^{k=n} \binom{n}{k} y^{n-k} z^k \partial_y^{n-k} \partial_z^k. \quad (3.24)$$

Then, a periodic solution \tilde{U}_2 is possible only if the following compatibility condition is satisfied.

$$\begin{aligned} & |\nabla_\perp P_1|^2 \langle \hat{U} \cdot DU_{0,\theta\theta} \rangle - (1/2) \ddot{\omega} (P_{1,x})^2 \langle \hat{U} \cdot U_{0,\theta} \rangle + P_{2,xx} \langle \hat{U} \cdot MU_{0,k} \rangle \\ & + \nabla^2 P_2 \langle \hat{U} \cdot DU_{0,\theta} \rangle - P_{2,t} \langle \hat{U} \cdot U_{0,\theta} \rangle = 0, \end{aligned} \quad (3.25)$$

where \hat{U} is an adjoint function of a null eigenfunction of L and $\langle \hat{U} \cdot U \rangle \equiv \int_0^{2\pi} (\hat{U} \cdot U) d\theta$.

A secular solution up to $O(\epsilon^2)$ is

$$\begin{aligned} U &= U_0(k, kx - \omega t + \phi) + \epsilon(P_1 + \epsilon P_2)U_{0,\theta} + (1/2)\epsilon^2 P_1^2 U_{0,\theta\theta} \\ &+ \epsilon(P_{1,x} + \epsilon P_{2,x})U_{0,k} + (1/2)\epsilon^2 (P_{1,x})^2 U_{0,kk} \\ &+ \epsilon^2 P_1 P_{1,x} U_{0,\theta k} \\ &= U_0(k + \tilde{\phi}_x, kx - \omega t + \tilde{\phi}), \end{aligned} \quad (3.26)$$

where $\tilde{\phi}$ is a renormalized phase defined by a renormalization transformation

$$\tilde{\phi}(x, \mathbf{r}_\perp, t) = \phi + \delta(x, \mathbf{r}_\perp, t) = \phi + \epsilon(P_1 + \epsilon P_2 + \cdots). \quad (3.27)$$

Since the renormalized phase should enjoy translational symmetry with respect to independent variables, (3.27) is rewritten as, shifting the origin $(x, \mathbf{r}_\perp, t) = (0, 0, 0)$ to an arbitrary point (x, \mathbf{r}_\perp, t) ,

$$\begin{aligned} \tilde{\phi}(t + \tau, x + \xi, \mathbf{r}_\perp + \boldsymbol{\eta}) &= \phi(x, \mathbf{r}_\perp, t) \\ &+ \epsilon\{\tilde{P}_1(\xi, \boldsymbol{\eta}; x, \mathbf{r}_\perp, t) + \epsilon\tilde{P}_2(\xi, \boldsymbol{\eta}, \tau; x, \mathbf{r}_\perp, t)\} \end{aligned} \quad (3.28)$$

where (x, \mathbf{r}_\perp, t) in polynomials P_1 and P_2 are replaced by $(\xi, \boldsymbol{\eta}, \tau)$ and their coefficients depend on the coordinate of the origin (x, \mathbf{r}_\perp, t) , e.g.

$$\tilde{P}_1(\xi, \boldsymbol{\eta}; x, \mathbf{r}_\perp, t) = P_{1,x}(x, \mathbf{r}_\perp, t)\xi + \nabla_\perp P_1(x, \mathbf{r}_\perp, t) \cdot \boldsymbol{\eta}$$

and so on. This reinterpretation of coefficients of secular terms is the key ingredient of the Lie approach of the RG method [6]. Hence, Eq.(3.28) with (3.19) and (3.23) reads

$$\begin{aligned} \tilde{\phi}_x &= [\partial_\xi(\epsilon\tilde{P}_1 + \epsilon^2\tilde{P}_2)]_0 = \epsilon P_{1,x}, \\ \nabla_\perp \tilde{\phi} &= [\partial_{\boldsymbol{\eta}}(\epsilon\tilde{P}_1 + \epsilon^2\tilde{P}_2)]_0 = \epsilon \nabla_\perp P_1, \\ \tilde{\phi}_{xx} &= [\partial_\xi^2(\epsilon^2\tilde{P}_2)]_0 = \epsilon^2 P_{2,xx}, \\ \nabla_\perp^2 \tilde{\phi} &= [\partial_{\boldsymbol{\eta}}^2(\epsilon^2\tilde{P}_2)]_0 = \epsilon^2 \nabla_\perp^2 P_2, \\ \tilde{\phi}_t &= [\partial_\tau(\epsilon^2\tilde{P}_2)]_0 = \epsilon^2 P_{2,t}, \end{aligned} \quad (3.29)$$

where $[f(\xi, \boldsymbol{\eta}, \tau; x, \mathbf{r}_\perp, t)]_0 = f(0, 0, 0; x, \mathbf{r}_\perp, t)$. Substituting these relations between differentials of the renormalized phase $\tilde{\phi}$ and the reinterpreted coefficients of polynomials P_1 and P_2 into (3.25), we obtain a non-isotropic Burgers (n-Burgers) equation:

$$\tilde{\phi}_t = D_\parallel \tilde{\phi}_{xx} + D_\perp \nabla_\perp^2 \tilde{\phi} + N_\parallel (\tilde{\phi}_x)^2 + N_\perp |\nabla_\perp \tilde{\phi}|^2, \quad (3.30)$$

and

$$\begin{aligned} D_\parallel &= D_\perp + D'_\parallel, \\ D_\perp &= \langle \hat{U} \cdot DU_{0,\theta} \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle, \quad D'_\parallel = \langle \hat{U} \cdot MU_{0,k} \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle \\ N_\perp &= \langle \hat{U} \cdot DU_{0,\theta\theta} \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle \\ N_\parallel &= N_\perp + \{ (1/2) \langle \hat{U} \cdot F'' : (U_{0,k})^2 \rangle + \langle \hat{U} \cdot MU_{0,\theta k} \rangle \} / \langle \hat{U} \cdot U_{0,\theta} \rangle, \\ &= -\ddot{\omega}/2, \end{aligned} \quad (3.31)$$

where the last equality of (3.31) comes from (3.6).

4 K-dV-Burgers Equation

In this section, it is assumed that the diffusion coefficient D_\parallel along the x direction in the n-Burgers equation (3.30) is small as ϵ :

$$D_\parallel \propto \langle \hat{U} \cdot (DU_{0,\theta} + MU_{0,k}) \rangle \sim O(\epsilon). \quad (4.32)$$

Nevertheless the net diffusion in the x direction is supposed to be much greater than that in the \mathbf{r}_\perp direction, i.e.

$$\nabla_\perp / \partial_x \sim O(\epsilon). \quad (4.33)$$

Eq.(4.32) implies that there is a periodic vector $V(\theta)$ such that

$$LV(\theta) = DU_{0,\theta} + MU_{0,k} + O(\epsilon). \quad (4.34)$$

Suppose that δ, \tilde{U} , and U are expanded as

$$\delta = \epsilon^2 \{ P_1(x, \mathbf{r}_\perp, t) + \epsilon P_2(x, \mathbf{r}_\perp, t) + \cdots \}, \quad (4.35)$$

$$\tilde{U} = \epsilon^3 \{ \tilde{U}_2(\theta) + \epsilon \tilde{U}_3(\theta, x, \mathbf{r}_\perp, t) + \cdots \}, \quad (4.36)$$

$$U = U_0(k, \theta) + \epsilon^2 (U_1 + \epsilon U_2 + \epsilon^2 U_3 + \cdots), \quad (4.37)$$

then the leading order perturbed terms ($O(\epsilon^2)$) gives the same equations as (3.16)–(3.18). As a polynomial solution of (3.18), we choose P_1 as

$$P_1 = P_{1,x}x, \quad P_{1,x} = \text{constant}, \quad (4.38)$$

instead of (3.19) since $|P_{1,x}| \gg |\nabla_\perp P_1|$ due to the assumption (4.33). To $O(\epsilon^3)$, we have

$$\tilde{L}U_2 = 0, \quad (4.39)$$

$$U_2 = P_2(x, \mathbf{r}_\perp, t)U_{0,\theta}(\theta) + P_{2,x}(x, \mathbf{r}_\perp, t)U_{0,k}(\theta) + \tilde{U}_2(\theta), \quad (4.40)$$

Substituting (4.40) into (4.39), we get

$$\begin{aligned} \tilde{L}\tilde{U}_2 = & P_{2,xx}(MU_{0,k} + DU_{0,\theta}) + \nabla_\perp^2 P_2 DU_{0,\theta} + \nabla^2 P_{2,x} DU_{0,k} \\ & - P_{2,t}U_{0,\theta} - P_{2,xt}U_{0,k}. \end{aligned} \quad (4.41)$$

Due to the consistent renormalization condition described in section 2 (i.e., consistent increasing of degrees of polynomial secular terms P_j), P_2 is a polynomial of degree two with respect to x and of degree one with respect to \mathbf{r}_\perp at most so that we can set $\nabla_\perp^2 P_2 = \nabla^2 P_{2,x} = 0$ and $P_{2,xt} = 0$, while

$$P_{2,xx} = \text{constant} = R_2. \quad (4.42)$$

If $P_{2,t} \neq 0$, the fourth term of the RHS of (4.41) is only a term which causes a secular behaviour of $\tilde{U}_2(\theta)$ with respect to θ and so $P_{2,t} = 0$. Then by virtue of (4.34) we have

$$\tilde{U}_2(\theta) = R_2 V(\theta), \quad (4.43)$$

and

$$P_2 = R_2 x^2/2 + \nabla_\perp P_2 \cdot \mathbf{r}_\perp, \quad \nabla_\perp P_2 = \text{constant}. \quad (4.44)$$

To $O(\epsilon^4)$ we have

$$\tilde{L}U_3 = (1/2)F'' : (U_1)^2, \quad (4.45)$$

$$\begin{aligned} U_3 = & P_3(x, \mathbf{r}_\perp, t)U_{0,\theta} + P_{3,x}(x, \mathbf{r}_\perp, t)U_{0,k} + (1/2)(P_1)^2 U_{0,\theta\theta} \\ & + P_1 P_{1,x} U_{0,\theta k} + (1/2)(P_{1,x})^2 U_{0,kk} \\ & + \tilde{U}_3(\theta, x, \mathbf{r}_\perp, t), \end{aligned} \quad (4.46)$$

from which we get

$$\begin{aligned} \tilde{L}\tilde{U}_3 = & P_{3,xx}MU_{0,k} + \nabla^2 P_3 DU_{0,\theta} - (1/2)(P_{1,x})^2 \ddot{\omega} U_{0,\theta} \\ & + \nabla^2 P_{3,x} DU_{0,k} - P_{3,t}U_{0,\theta} - P_{3,xt}U_{0,k}. \end{aligned} \quad (4.47)$$

In view of (4.38) and (4.44), the consistent renormalization condition requires that P_3 contains x^3 and the first term of the RHS of (4.46) is secular with respect to x . Therefore we set

$$\tilde{U}_3(\theta, x, \mathbf{r}_\perp, t) = R_3(x)V(\theta) + \bar{U}_3(\theta), \quad (4.48)$$

where

$$R_3 = R_{3,xx}x, \quad R_{3,x} = \text{constant}, \quad (4.49)$$

and (4.47) is rewritten as

$$\begin{aligned} L\bar{U}_3(\theta) = & -(R_3 - P_{3,xx})(MU_{0,k} + DU_{0,\theta}) + \nabla_\perp^2 P_3 DU_{0,\theta} - (1/2)(P_{1,x})^2 \ddot{\omega} U_{0,\theta} \\ & - R_{3,x} MV(\theta) + \nabla^2 P_{3,x} DU_{0,k} - P_{3,t} U_{0,\theta} - P_{3,xt} U_{0,k}. \end{aligned} \quad (4.50)$$

Since (4.50) is an equation to $\bar{U}_3(\theta)$, all the coefficients of functions of θ in the RHS of (4.50) do not depend on (x, \mathbf{r}_\perp, t) , that is,

$$\begin{aligned} R_3 - P_{3,xx} &= c_1, \quad \nabla_\perp^2 P_3 = c_2, \\ \nabla^2 P_{3,x} &= c_3, \quad P_{3,t} = c_4, \quad P_{3,xt} = c_5, \end{aligned} \quad (4.51)$$

where c_n are arbitrary constants. The consistent renormalization condition yields $c_2 = c_5 = 0$, $c_3 = P_{3,xxx}$ and $c_1 = 0$ so that a secular coefficient $R_3(x)$ is consistently removed by a renormalization transformation to ϕ (see (4.59)). Thus (4.51) gives

$$P_3 = P_{3,xxx}x^3/3! + x(\mathbf{r}_\perp : \nabla_\perp)P_3 + P_{3,t}t, \quad (4.52)$$

The compatibility condition for a periodic solution $\bar{U}_3(\theta)$ requires that

$$\begin{aligned} & P_{2,xx}(\langle \hat{U} \cdot MU_{0,k} \rangle + \langle \hat{U} \cdot DU_{0,\theta} \rangle)/\epsilon - (\ddot{\omega}/2)(P_{1,x})^2 \langle \hat{U} \cdot U_{0,\theta} \rangle \\ & + P_{3,xxx}(\langle \hat{U} \cdot MV \rangle + \langle \hat{U} \cdot DU_{0,k} \rangle) - P_{3,t} \langle \hat{U} \cdot U_{0,\theta} \rangle = 0. \end{aligned} \quad (4.53)$$

Here the first term in the LHS of (4.53) comes from $O(\epsilon^3)$ terms. Now we arrive at a renormalization transformation to $O(\epsilon^4)$

$$\tilde{\phi}(x, \mathbf{r}_\perp, t) = \phi + \epsilon^2\{P_1(x) + \epsilon P_2(x, \mathbf{r}_\perp) + \epsilon^2 P_3(x, \mathbf{r}_\perp, t)\}. \quad (4.54)$$

Following the same procedure as in the case of the n-Burgers equation, we obtain from (4.54), (4.38), (4.44) and (4.52)

$$\tilde{\phi}_x = \epsilon^2 P_{1,x}, \quad \tilde{\phi}_{xx} = \epsilon^3 P_{2,xx}, \quad \tilde{\phi}_{xxx} = \epsilon^4 P_{3,xxx}, \quad \tilde{\phi}_t = \epsilon^4 P_{3,t}. \quad (4.55)$$

Substituting (4.55) into (4.53), we arrive at the K-dV-Burgers (K-B) equation:

$$\tilde{\phi}_t = D_\parallel \tilde{\phi}_{xx} + N_\parallel (\tilde{\phi}_x)^2 + A \tilde{\phi}_{xxx}, \quad (4.56)$$

where

$$\begin{aligned} A &= B + A', \\ B &= \langle \hat{U} \cdot MV \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle, \quad A' = \langle \hat{U} \cdot DU_{0,k} \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle. \end{aligned} \quad (4.57)$$

Notice that secular terms in \tilde{U} are also eliminated by the present renormalization procedure. To $O(\epsilon^4)$, (4.43) and (4.48) give

$$\tilde{U} = \epsilon^3 \{ R_2 + \epsilon R_3(x) \} V(\theta) + \epsilon^4 \bar{U}_3(\theta), \quad (4.58)$$

where a secular term R_3 is removed by introducing a renormalized R_2 such that

$$\tilde{R}_2(x, \mathbf{r}_\perp, t) = R_2 + \epsilon R_3(x). \quad (4.59)$$

This renormalization transformation is found to be consistent with that to the phase ϕ (4.54), since (4.42), (4.55) and (4.51) with $c_1 = 0$ imply $\tilde{R}_2 = \tilde{\phi}_{xx}/\epsilon^3$.

If we introduce the following expansion

$$\begin{aligned} \delta &= \epsilon^4 \{ P_1 + \epsilon P_2 + \cdots \}, \\ \tilde{U} &= \epsilon^5 \{ \tilde{U}_2 + \epsilon \tilde{U}_3 + \cdots \}, \\ U &= U_0(k, \theta) + \epsilon^4 (U_1 + \epsilon U_2 + \epsilon^2 U_3 + \cdots), \end{aligned}$$

instead of (4.35), (4.36), (4.37), and an auxiliary condition $D_\parallel \sim O(\epsilon^2)$ instead of (4.32), the similar procedure as above yields, to $O(\epsilon^8)$, the following K-dV-Burgers equation with the perpendicular diffusion term

$$\tilde{\phi}_t = D_\parallel \tilde{\phi}_{xx} + D_\perp \nabla_\perp^2 \tilde{\phi} + N_\parallel (\tilde{\phi}_x)^2 + A \tilde{\phi}_{xxx}, \quad (4.60)$$

5 Non-isotropic Kuramoto-Sivashinsky Equation

The diffusion coefficient D_\perp in the n-Burgers equation (3.30) is assumed to be small so that

$$D_\perp \propto \langle \hat{U} \cdot DU_{0,\theta} \rangle \sim O(\epsilon^2). \quad (5.61)$$

and net diffusions along both x and \mathbf{r}_\perp directions are nevertheless the same order of magnitude, that is,

$$\partial_x^2 / \nabla_\perp^2 \sim O(\epsilon^2). \quad (5.62)$$

Eq.(5.61) guarantees existence of a periodic vector $W(\theta)$ such that

$$LW(\theta) = DU_{0,\theta} + O(\epsilon^2). \quad (5.63)$$

Suppose that δ, \tilde{U} , and U are expanded as

$$\delta = \epsilon^3 \{P_1(x, \mathbf{r}_\perp, t) + \epsilon P_2(x, \mathbf{r}_\perp, t) + \dots\}, \quad (5.64)$$

$$\tilde{U} = \epsilon^4 \{\tilde{U}_2(\theta) + \epsilon \tilde{U}_3(\theta, x, \mathbf{r}_\perp, t) + \dots\}, \quad (5.65)$$

$$U = U_0(k, \theta) + \epsilon^3(U_1 + \epsilon U_2 + \epsilon^2 U_3 + \dots), \quad (5.66)$$

then perturbed equations up to $O(\epsilon^5)$ give the same equations as (4.39) and (4.40) for U_j ($j = 1, 2, 3$).

$$\tilde{L}U_j = 0, \quad (5.67)$$

$$U_j = P_j(x, \mathbf{r}_\perp, t)U_{0,\theta}(\theta) + P_{j,x}(x, \mathbf{r}_\perp, t)U_{0,k}(\theta) + \tilde{U}_j, \quad (5.68)$$

Noting $\tilde{U}_1 = 0$ and following (3.16), (3.17), and (3.18), we have (3.19) with $P_{1,x} = 0$ due to the assumption (5.62), that is,

$$U_1 = P_1(\mathbf{r}_\perp)U_{0,\theta}, \quad P_1 = \nabla_\perp P_1 \cdot \mathbf{r}_\perp. \quad (5.69)$$

For $j = 2$, substituting (5.68) into (5.67), we have

$$\begin{aligned} L\tilde{U}_2(\theta) &= \nabla_\perp^2 P_2 DU_{0,\theta} + P_{2,xx}(MU_{0,k} + DU_{0,\theta}) + \nabla^2 P_{2,x} DU_{0,k} \\ &\quad - P_{2,t}U_{0,\theta} - P_{2,xt}U_{0,k}, \end{aligned} \quad (5.70)$$

Since \tilde{U}_2 is a function of θ only, all the coefficients of functions of θ in the RHS of (5.70) must be constant. The consistent renormalization condition and (5.69) implies that a degree of polynomial P_2 should be two with respect to \mathbf{r}_\perp and one with respect to x and t at most, i.e.

$$P_{2,xx} = \nabla^2 P_{2,x} = P_{2,xt} = 0.$$

The first term of the RHS of (5.70) does not cause a secular behaviour of \tilde{U}_2 owing to (5.63) and so $P_{2,t} = 0$ is necessary for a periodic solution \tilde{U}_2 . Thus we have

$$\tilde{U}_2(\theta) = S_2 W(\theta), \quad S_2 = \nabla_\perp^2 P_2 = \text{constant}, \quad (5.71)$$

$$P_2 = P_{2,x}x + (\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp) P_2 / 2. \quad (5.72)$$

where all the coefficients of monomials in (5.72) are arbitrary constants.

For $j = 3$, we have the same equation as (5.70), where the suffix 2 is replaced by 3.

$$\begin{aligned} L\tilde{U}_3(\theta) &= \nabla_\perp^2 P_3 DU_{0,\theta} + P_{3,xx}(MU_{0,k} + DU_{0,\theta}) + \nabla^2 P_{3,x} DU_{0,k} \\ &\quad - P_{3,t}U_{0,\theta} - P_{3,xt}U_{0,k}, \end{aligned} \quad (5.73)$$

The consistent renormalization condition implies that a degree of polynomial P_3 should be three with respect to \mathbf{r}_\perp and so $\nabla_\perp^2 P_3$ in the first term of the RHS of (5.73) is secular with respect to \mathbf{r}_\perp , while

$$P_{3,xx} = 0, \quad \nabla^2 P_{3,x} = 0, \quad P_{3,xt} = 0. \quad (5.74)$$

Therefore, \tilde{U}_3 takes the form

$$\tilde{U}_3 = S_3(\mathbf{r}_\perp)W(\theta) + \bar{U}_3(\theta), \quad (5.75)$$

where $S(\mathbf{r}_\perp)$ is a polynomial of degree one with respect to \mathbf{r}_\perp . Then, (5.73) reads

$$L\bar{U}_3(\theta) = (\nabla_\perp^2 P_3 - S_3)DU_{0,\theta} - P_{3,t}U_{0,\theta}. \quad (5.76)$$

Here $P_{3,t} = 0$ is again necessary for a periodic solution $U_3(\theta)$ and $\nabla_\perp^2 P_3 - S_3 = 0$ so that a secular term S_3 is automatically eliminated as soon as the phase is renormalized as shown in the last paragraph of this section. Now, we set $\bar{U}_3(\theta) = 0$ without loss of generality and have

$$S_3 = \nabla_\perp^2 P_3 = \nabla_\perp S_3 \cdot \mathbf{r}_\perp, \quad (5.77)$$

$$P_3 = x(\mathbf{r}_\perp : \nabla_\perp)P_{3,x} + (\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp)P_3/3!, \quad (5.78)$$

where $\nabla_\perp S_3$, $\nabla_\perp P_{3,x}$ and $\nabla_\perp \nabla_\perp \nabla_\perp P_3$ are arbitrary constants. A nonlinear term enters in the perturbed equation to $O(\epsilon^6)$:

$$\tilde{L}U_4 = (1/2)F'' : U_1^2, \quad (5.79)$$

$$U_4 = P_4(x, \mathbf{r}_\perp, t)U_{0,\theta} + P_{4,x}(x, \mathbf{r}_\perp, t)U_{0,k} + (1/2)P_1^2(\mathbf{r}_\perp)U_{0,\theta\theta} + \tilde{U}_4(\theta, x, \mathbf{r}_\perp, t), \quad (5.80)$$

from which, we get

$$\begin{aligned} \tilde{L}\tilde{U}_4 &= \nabla^2 P_4 DU_{0,\theta} + P_{4,xx} MU_{0,k} + \nabla^2 P_{4,x} DU_{0,k} \\ &\quad + (1/2)\nabla^2 P_1^2 DU_{0,\theta\theta} - P_{4,t}U_{0,\theta} - P_{4,xt}U_{0,k}. \end{aligned} \quad (5.81)$$

The similar discussion leading to (5.75) yields

$$\tilde{U}_4(\theta, x, \mathbf{r}_\perp, t) = S_4(x, \mathbf{r}_\perp)W(\theta) + \bar{U}_4(\theta), \quad (5.82)$$

where

$$S_4 = S_{4,x}x + (\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp)S_4/2, \quad (5.83)$$

and $S_{4,x}$ and $\nabla_\perp \nabla_\perp S_4$ are arbitrary constants. Then (5.81) is rewritten as

$$\begin{aligned} L\bar{U}_4(\theta) &= -(S_4 - \nabla^2 P_4)DU_{0,\theta} + P_{4,xx} MU_{0,k} + S_{4,x} MW \\ &\quad + \nabla^2 P_{4,x} DU_{0,k} + (1/2)\nabla^2 P_1^2 DU_{0,\theta\theta} + \nabla^2 S_4 DW \\ &\quad - P_{4,t}U_{0,\theta} - P_{4,xt}U_{0,k}, \end{aligned}$$

which is an equation for $\bar{U}_4(\theta)$ and all the coefficients of functions of θ should be constant.

$$S_4 - \nabla^2 P_4 = c_1, \quad P_{4,xx} = c_2, \quad \nabla^2 P_{4,x} = c_2, \quad P_{4,t} = c_4, \quad (5.84)$$

and $c_5 = P_{4,xt} = 0$ holds again. The compatibility condition for $\bar{U}_4(\theta)$ requires

$$\begin{aligned} & (\nabla^2 P_2 / \epsilon^2) \langle \hat{U} \cdot DU_{0,\theta} \rangle + P_{4,xx} \langle \hat{U} \cdot MU_{0,k} \rangle + \nabla_\perp^2 P_{4,x} \langle \hat{U} \cdot MW(\theta) \rangle \\ & + \nabla_\perp^2 P_{4,x} \langle \hat{U} \cdot DU_{0,k} \rangle + |\nabla_\perp P_1|^2 \langle \hat{U} \cdot DU_{0,\theta\theta} \rangle + \nabla_\perp^4 P_4 \langle \hat{U} \cdot DW(\theta) \rangle \\ & - P_{4,t} \langle \hat{U} \cdot U_{0,\theta} \rangle = 0. \end{aligned} \quad (5.85)$$

A renormalization transformation to $O(\epsilon^6)$ is

$$\tilde{\phi}(x, \mathbf{r}_\perp, t) = \phi + \epsilon^3 \{P_1 + \epsilon P_2 + \epsilon^2 P_3 + \epsilon^3 P_4\}, \quad (5.86)$$

which gives, noting (5.69), (5.72), (5.78), and (5.84),

$$\begin{aligned} \nabla_\perp \tilde{\phi} &= \epsilon^3 \nabla_\perp P_1, \quad \tilde{\phi}_{xx} = \epsilon^6 P_{4,xx}, \quad \nabla_\perp^2 \tilde{\phi} = \epsilon^4 \nabla_\perp^2 P_2, \\ \nabla_\perp^2 \tilde{\phi}_x &= \epsilon^6 \nabla_\perp^2 P_{4,x}, \quad \nabla_\perp^4 \tilde{\phi} = \epsilon^6 \nabla_\perp^4 P_4, \quad \tilde{\phi}_t = \epsilon^6 P_{4,t}. \end{aligned} \quad (5.87)$$

Substituting (5.87) into (5.85), we arrive at a non-isotropic Kuramoto-Sivashinsky (n-K-S) equation:

$$\tilde{\phi}_t = D_\parallel' \tilde{\phi}_{xx} + D_\perp \nabla_\perp^2 \tilde{\phi} + N_\perp |\nabla_\perp \tilde{\phi}|^2 + E \nabla_\perp^4 \tilde{\phi} + G \nabla_\perp^2 \tilde{\phi}_x, \quad (5.88)$$

where

$$\begin{aligned} E &= \langle \hat{U} \cdot DW(\theta) \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle, \\ G &= A' + H \\ A' &= \langle \hat{U} \cdot DU_{0,k} \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle, \quad H = \langle \hat{U} \cdot MW(\theta) \rangle / \langle \hat{U} \cdot U_{0,\theta} \rangle. \end{aligned} \quad (5.89)$$

Secular terms in \tilde{U} are also eliminated by the present renormalization procedure. To $O(\epsilon^6)$, (5.71), (5.75), and (5.82) give

$$\tilde{U} = \epsilon^4 \{S_2 + \epsilon S_3(\mathbf{r}_\perp) + \epsilon^2 S_4(x, \mathbf{r}_\perp)\} W(\theta) + \epsilon^6 \bar{U}_4(\theta), \quad (5.90)$$

where secular terms S_3 and S_4 are removed by introducing a renormalized S_2 such that

$$\tilde{S}_2(x, \mathbf{r}_\perp, t) = S_2 + \epsilon S_3(\mathbf{r}_\perp) + \epsilon^2 S_4(x, \mathbf{r}_\perp). \quad (5.91)$$

This renormalization transformation is found to be consistent with that to the phase ϕ (5.86), since (5.71), (5.77), (5.87), and (5.84) with $c_1 = 0$ imply $\tilde{S}_2 = \nabla_\perp^2 \tilde{\phi} / \epsilon^4$.

6 Benney Equation in Three Dimension

In addition to the assumption (5.61), the wave number k is also assumed to be as small as $O(\epsilon)$. Furthermore it may be reasonable to assume that $\omega(k)$ and $U_0(k, \theta)$ are functions of k^2 , which is satisfied in the case of the complex Ginzburg-Landau equation analyzed in the next section. Then the following estimates hold

$$\dot{\omega} \sim O(\epsilon), \quad U_{0,k} \sim O(\epsilon), \quad (6.92)$$

and

$$A' \propto \langle \hat{U} \cdot DU_{0,k} \rangle \sim O(\epsilon), \quad B \propto \langle \hat{U} \cdot MV(\theta) \rangle \sim O(\epsilon), \quad (6.93)$$

$$D'_{\parallel} \propto \langle \hat{U} \cdot MU_{0,k} \rangle \sim O(\epsilon^2), \quad (6.94)$$

or

$$A = A' + B \sim O(\epsilon), \quad D_{\parallel} = D_{\perp} + D'_{\parallel} \sim O(\epsilon^2), \quad (6.95)$$

namely both coefficients of diffusion D_{\perp} and D_{\parallel} are as small as $O(\epsilon^2)$ while the coefficient of dispersion A is $O(\epsilon)$. Here, the linearized operator L and \tilde{L} are rearranged as

$$L = L' - k^2 D \partial_{\theta}^2, \quad (6.96)$$

$$L' = -\omega \partial_{\theta} - F'(U_0), \quad (6.97)$$

and

$$\tilde{L} = \tilde{L}' - M \partial_x - k^2 D \partial_{\theta}^2, \quad (6.98)$$

$$\tilde{L}' = \partial_t + L' + \nabla^2 D. \quad (6.99)$$

Then (3.3)–(3.7) become

$$L'U_{0,\theta} = k^2 D \partial_{\theta}^2 U_{0,\theta}, \quad (6.100)$$

$$L'U_{0,\theta\theta} = F'' : U_{0,\theta}^2 + k^2 D \partial_{\theta}^2 U_{0,\theta\theta}, \quad (6.101)$$

$$L'U_{0,k} = MU_{0,\theta} + k^2 D \partial_{\theta}^2 U_{0,k}, \quad (6.102)$$

$$\begin{aligned} L'U_{0,kk} &= F'' : U_{0,k}^2 + 2MU_{0,\theta k} + 2DU_{0,\theta\theta} \\ &\quad + \ddot{\omega}U_{0,\theta} + k^2 D \partial_{\theta}^2 U_{0,kk}, \end{aligned} \quad (6.103)$$

$$L'U_{0,\theta k} = MU_{0,\theta\theta} + F'' : U_{0,\theta}U_{0,k} + k^2 D \partial_{\theta}^2 U_{0,\theta k}. \quad (6.104)$$

Suppose that δ, \tilde{U} , and U are expanded in the same forms as (5.64), (5.65), and (5.66), then up to $O(\epsilon^5)$ perturbed secular fields U_j ($j = 1, 2, 3$) obey

$$\tilde{L}'U_j = M \partial_x U_{j-1}, \quad (6.105)$$

$$U_j = P_j(x, \mathbf{r}_{\perp}, t)U_{0,\theta} + P_{j-1,x}U_{0,k} + \tilde{U}_j, \quad (6.106)$$

where $P_0 = \tilde{U}_1 = 0$.

For $j = 1$, (6.105) and (6.106) give

$$P_{1,t}U_{0,\theta} - \nabla^2 P_1 D U_{0,\theta} = 0, \quad (6.107)$$

which implies

$$P_{1,t} = 0, \quad \nabla^2 P_1 = 0, \quad (6.108)$$

and

$$P_1 = P_{1,x}x + \nabla_\perp P_1 \cdot \mathbf{r}_\perp. \quad (6.109)$$

To $j = 2$, (6.105) and (6.106) give

$$\tilde{L}'\tilde{U}_2 = -P_{2,t}U_{0,\theta} + \nabla^2 P_2 D U_{0,\theta}. \quad (6.110)$$

The similar discussion as in the previous sections yields

$$\tilde{U}_2 = S_2 W(\theta), \quad S_2 = \nabla^2 P_2 = \text{constant}, \quad P_{2,t} = 0, \quad (6.111)$$

and

$$P_2 = P_{2,xx}x^2/2 + x(\mathbf{r}_\perp : \nabla_\perp)P_{2,x} + (\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp)P_2/2, \quad (6.112)$$

where $P_{2,xx}$, $\nabla_\perp P_{2,x}$ and $(\nabla_\perp)^2 P_2$ are arbitrary constants.

Similarly, to $j = 3$, we have

$$\tilde{U}_3 = S_3 W(\theta), \quad S_3 = \nabla^2 P_3 = S_{3,x}x + \nabla_\perp S_3 \cdot \mathbf{r}_\perp, \quad P_{3,t} = 0, \quad (6.113)$$

and

$$\begin{aligned} P_3 = & P_{3,xxx}x^3/3! + x^2(\mathbf{r}_\perp : \nabla_\perp)P_{3,xx}/2 + x(\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp)P_{3,x}/2 \\ & + (\mathbf{r}_\perp \mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp \nabla_\perp)P_3/3!, \end{aligned} \quad (6.114)$$

which is a polynomial of degree three and all the coefficients are arbitrary constants

To $O(\epsilon^6)$, a nonlinear term appears as

$$\tilde{L}'U_4 = (1/2)F'' : U_1^2 + M\partial_x U_3, \quad (6.115)$$

$$U_4 = P_4 U_{0,\theta} + P_{3,x} U_{0,k} + (1/2)P_1^2 U_{0,\theta\theta} + \tilde{U}_4(\theta, x, \mathbf{r}_\perp, t). \quad (6.116)$$

Substituting (6.116) into (6.115), we get

$$\begin{aligned} \tilde{L}'\tilde{U}_4 = & \nabla^2 P_4 D U_{0,\theta} + \nabla^2 P_{3,x} D U_{0,k} + P_{2,xx} M U_{0,k} \\ & + \nabla^2 (P_1^2/2) D U_{0,\theta\theta} - P_{4,t} U_{0,\theta}. \end{aligned} \quad (6.117)$$

Let set

$$\tilde{U}_4(\theta) = S_4 W(\theta) + \bar{U}_4(\theta), \quad (6.118)$$

where

$$S_4 = S_{4,xx}x^2/2 + x(\mathbf{r}_\perp : \nabla_\perp)S_{4,x} + (\mathbf{r}_\perp \mathbf{r}_\perp : \nabla_\perp \nabla_\perp)S_4/2, \quad (6.119)$$

which is a polynomial of degree two, then we have

$$\begin{aligned} L'\bar{U}_4 = & -(S_4 - \nabla^2 P_4)DU_{0,\theta} + \nabla^2 P_{3,x}DU_{0,k} + P_{2,xx}MU_{0,k} \\ & - \nabla^2 S_4 DW - S_{3,x}MW + \nabla^2(P_1^2/2)DU_{0,\theta} - P_{4,t}U_{0,\theta}. \end{aligned} \quad (6.120)$$

This equation is valid only when all the coefficients of functions of θ in (6.120) are constant and the compatibility condition gives

$$\begin{aligned} & (\nabla^2 P_2/\epsilon^2)\langle \hat{U} \cdot DU_{0,\theta} \rangle + P_{2,xx}\langle \hat{U} \cdot MU_{0,k} \rangle + \nabla^2 P_{3,x}\langle \hat{U} \cdot MW \rangle \\ & + \nabla^2 P_{3,x}\langle \hat{U} \cdot DU_{0,k} \rangle + |\nabla P_1|^2\langle \hat{U} \cdot DU_{0,\theta\theta} \rangle + \nabla^4 P_4\langle \hat{U} \cdot DW \rangle \\ & - P_{4,t}\langle \hat{U} \cdot U_{0,\theta} \rangle = 0. \end{aligned} \quad (6.121)$$

Now a renormalization transformation to $O(\epsilon^6)$ takes the same form as (5.86). From (5.86), (6.109), (6.112), (6.114) and (6.120) we obtain

$$\begin{aligned} \tilde{\phi}_x &= \epsilon^3 P_{1,x}, \quad \nabla_\perp \tilde{\phi} = \epsilon^3 \nabla_\perp P_1, \quad \tilde{\phi}_{xx} = \epsilon^4 P_{2,xx}, \quad \nabla_\perp^2 \tilde{\phi} = \epsilon^4 \nabla_\perp^2 P_2, \\ \tilde{\phi}_{xxx} &= \epsilon^5 P_{3,xxx}, \quad \nabla_\perp^2 \tilde{\phi}_x = \epsilon^5 \nabla_\perp^2 P_{3,x}, \quad \tilde{\phi}_{xxxx} = \epsilon^6 P_{4,xxxx}, \quad \nabla_\perp^4 \tilde{\phi} = \epsilon^6 \nabla_\perp^4 P_4, \\ \tilde{\phi}_t &= \epsilon^6 P_{4,t}. \end{aligned} \quad (6.122)$$

Substituting (6.122) into (6.121), we arrive at a generalized Benney equation in the three dimensional space:

$$\tilde{\phi}_t = D_\parallel \tilde{\phi}_{xx} + D_\perp \nabla_\perp^2 \tilde{\phi} + G(\tilde{\phi}_{xxx} + \nabla_\perp^2 \tilde{\phi}_x) + E \nabla^4 \tilde{\phi} + N_\perp |\nabla \tilde{\phi}|^2. \quad (6.123)$$

As in the case of the n-K-S equation, secular terms in \tilde{U} are also automatically removed when ϕ is renormalized.

7 Complex Ginzburg-Landau Equation

As an application of the previous results, let us calculate explicitly various coefficients of the phase equations for the complex Ginzburg-Landau (cGL) equation.

$$\begin{cases} \Psi_t = \gamma \Psi - \beta |\Psi|^2 \Psi + \alpha \nabla^2 \Psi \\ \bar{\Psi}_t = \gamma \bar{\Psi} - \bar{\beta} |\bar{\Psi}|^2 \bar{\Psi} + \bar{\alpha} \nabla^2 \bar{\Psi}, \end{cases} \quad (7.124)$$

where Ψ is a complex variable, γ is a real constant, α and β are complex constants. The bar denotes complex conjugation. Setting

$$U = \begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix}, \quad F = \begin{pmatrix} \gamma\Psi - \beta|\Psi|^2\Psi \\ \gamma\bar{\Psi} - \bar{\beta}|\Psi|^2\bar{\Psi} \end{pmatrix}, \quad D = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \quad (7.125)$$

the cGL equation is transformed into the standard form of a reaction-diffusion system (3.1). As a periodically oscillating solution of (7.124), we take

$$U_0 = \begin{pmatrix} \Psi_0 \\ \bar{\Psi}_0 \end{pmatrix} = a(k) \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}, \quad (7.126)$$

where a real amplitude $a(k)$ and a frequency $\omega(k)$ satisfy the following dispersion relation

$$-i\omega = \gamma - \beta a^2 - \alpha k^2, \quad (7.127)$$

which $\omega(k)$ and $a(k)$ are functions of k^2 as speculated in section 6. Since the adjoint operator L^\dagger of L defined in (3.3) is

$$L^\dagger = \omega\partial_\theta - F'^\dagger(U_0) - D^\dagger k^2 \partial_\theta^2, \quad (7.128)$$

where

$$F'^\dagger(U_0) = \begin{pmatrix} \gamma - 2\bar{\beta}a^2 & -\beta\Psi_0^2 \\ -\bar{\beta}\bar{\Psi}_0^2 & \gamma - 2\beta a^2 \end{pmatrix}, \quad D^\dagger = \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}, \quad (7.129)$$

we explicitly find the null vector of L^\dagger as

$$\hat{U} = i/2\beta'a \begin{pmatrix} \beta e^{i\theta} \\ -\bar{\beta} e^{-i\theta} \end{pmatrix}, \quad (7.130)$$

with the normalization $\langle \hat{U} \cdot U_{0,\theta} \rangle = 1$ and $\beta = \beta' + i\beta''$. Various differentials of U_0 are also explicitly given by

$$\begin{aligned} U_{0,\theta} &= a \begin{pmatrix} ie^{i\theta} \\ -ie^{-i\theta} \end{pmatrix}, \quad U_{0,k} = \dot{a} \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}, \quad U_{0,\theta\theta} = -a \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}, \\ U_{0,\theta k} &= \dot{a} \begin{pmatrix} ie^{i\theta} \\ -ie^{-i\theta} \end{pmatrix}, \quad U_{0,kk} = \ddot{a} \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}. \end{aligned} \quad (7.131)$$

Using (7.130) and (7.131), the coefficients of diffusion and nonlinearity are calculated as

$$D_\perp = \langle \hat{U} \cdot DU_{0,\theta} \rangle = \text{Re}(\alpha\bar{\beta})/\beta', \quad (7.132)$$

$$\begin{aligned} D'_\parallel &= \langle \hat{U} \cdot MU_{0,k} \rangle = (-\dot{a}/\beta'a)\{\dot{\omega}\beta'' - 2k\text{Re}(\alpha\bar{\beta})\} \\ &= -2\dot{a}^2|\beta|^2/\beta', \end{aligned} \quad (7.133)$$

$$N_\perp = \langle \hat{U} \cdot DU_{0,\theta\theta} \rangle = -\text{Im}(\alpha\bar{\beta})/\beta' = -\ddot{\omega}/2 = N_\parallel, \quad (7.134)$$

where N_{\parallel} happens to be identical with N_{\perp} in this case. The coefficient of dispersion in the K-dV-Burgers equation is calculated as follows. The assumption (4.32) reads

$$D_{\parallel} = \{Re(\alpha\bar{\beta}) - 2\dot{a}^2|\beta|^2\}/\beta' \sim O(\epsilon), \quad (7.135)$$

and

$$Re(\alpha\bar{\beta}) \sim 2\dot{a}^2|\beta|^2. \quad (7.136)$$

Then V in (4.34) is obtained as

$$V(\theta) = (Im(\bar{\alpha}\beta)/2a\dot{a}|\beta|^2)U_{0,k}, \quad (7.137)$$

and we have the coefficient of dispersion

$$A' = \langle \hat{U} \cdot DU_{0,k} \rangle = \dot{a}Im(\alpha\bar{\beta})/a\beta', \quad (7.138)$$

$$\begin{aligned} B &= \langle \hat{U} \cdot MV(\theta) \rangle = (Im(\bar{\alpha}\beta)/2a\dot{a}|\beta|^2)\langle \hat{U} \cdot MU_{0,k} \rangle \\ &= -\dot{a}Im(\bar{\alpha}\beta)/a\beta', \end{aligned} \quad (7.139)$$

$$A = B + A' = 2\dot{a}Im(\alpha\bar{\beta})/a\beta'. \quad (7.140)$$

From the Kuramoto-Sivashinsky scale (5.61), we have

$$D_{\perp} = \langle \hat{U} \cdot DU_{0,\theta} \rangle = Re(\alpha\bar{\beta})/\beta' \sim Re(\alpha\bar{\beta}) \sim O(\epsilon^2). \quad (7.141)$$

Then W in (5.63) is found to have the same expression as V since the forcing terms in Eqs.(4.34) and (5.63) have the same non-secular component under the conditions (7.135) and (7.141).

The coefficients of the K-S equation are obtained as

$$\begin{aligned} E &= \langle \hat{U} \cdot DW(\theta) \rangle = (Im(\bar{\alpha}\beta)/2a\dot{a}|\beta|^2)\langle \hat{U} \cdot DU_{0,k} \rangle \\ &= -(Im(\bar{\alpha}\beta))^2/2a^2\beta'|\beta|^2, \end{aligned} \quad (7.142)$$

$$H = \langle \hat{U} \cdot MW(\theta) \rangle = B, \quad (7.143)$$

$$G = A' + H = A' + B = A. \quad (7.144)$$

All the coefficients of the Benney equation are given in terms of those of the K-B equation and the K-S equation. The isotropic part of coefficients of the n-Burgers equation and the K-S equation agrees with the result in [2].

8 Concluding Remarks

Let us compare the present derivation of phase equations with a possible derivation by means of the reductive perturbation (RP) method [7] or the

multi space-time scale method, although the latter derivation has not been accomplished yet. The initial setting of perturbation (3.8) (and (3.9)) and auxiliary conditions (4.32) and (5.61) are same for both derivations. In addition to the initial setting and the auxiliary condition, the RG method assumes a naive expansion of secular solutions such as (3.10)–(3.12) and (4.35)–(4.37) etc. . Then straightforward calculations of secular terms with the aid of the consistent renormalization condition and the RG procedure of the Lie approach lead to the final results. The type of derived phase equations depends only on a specific form of expansion of secular terms and the auxiliary condition. On the other hand, the RP method would requires specific scalings for not only perturbed fields but also independent variables, which are available after a derived equation is set up. It should be mentioned that the present RG method relies on an explicit secular solution although the RP method would not. However, the step to obtain an explicit secular solution may be largely skipped as far as the final results are concerned or possibly by the proto-RG approach developed recently [8].

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